ON THE EQUATIONS OF MOTION OF SYSTEMS WITH NONLINEAR, NONHOLONOMIC CONSTRAINTS

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The equations of motion of a system with a nonlinear, nonholonomic constraint as derived by Hamel do not describe the behavior of the system, if we consider this system a limiting case of a nonholonomic system with linear constraints.

The derivation of the equations of motion of mechanical systems with nonlinear, nonholonomic constraints is presented in a number of papers [1] to [4]. A systematic development of the methods of analytical mechanics for nonholonomic systems with nonlinear constraints has been presented by Johnsen [1], Hamel [4], Chetaev [3] and V.S. Novoselov*. There has been in the past a lively discussion between Appell [2], Delassus [5], Beghin [6] and others on the realization of nonlinear, nonholonomic constraints, but in spite of that none of the papers contains examples of systems with nonlinear, nonholonomic constraints which would differ essentially from the example presented by Appell in 1911.

A number of authors [4] to [6] became interested in the example of Appell. This example has been investigated in detail by Hamel who derived for it equations of motion, starting from the generally accepted definition of virtual displacements for systems with nonlinear, nonholonomic constraints.

In this paper we have demostrated that a more correct approach to the system in the example of Appell-Hamel leads to motions which are not described by the equations of motion derived by Hamel.

1. The example of Appell-Hamel. The equations of motion and formulation of the problem. Appell [2] and Hamel consider a nonholonomic system shown in Fig. 1. The weight of mass m hangs on a thread drawn through pulleys and wound on a drum of radius b. The drum is joined rigidly to a wheel of radius a , which rolls without sliding on a horizontal plane touching it at the point B . The legs of the frame supporting the pulleys and keeping the wheel vertical, slide on the horizontal plane without friction. Let X,Y be the coordinates of the point B (the point of contact), θ be the angle

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* V.S. Novoselov, Certain problems of nonholonomic mechanics, Doctoral dissertation, Moscow, 1958.

between the plane of the wheel and the x-axis (Fig. 1), φ the angle of spin of the wheel, x,y,z the coordinates of the mass m. From Fig. 1 we have

$$dz = bd\varphi \,. \qquad (b>0) \tag{1.1}$$

The coordinates X,Y and x,y are related by

$$x = X + \rho\cos\theta, \qquad y = Y + \rho\sin\theta \tag{1.2}$$

The condition for rolling without sliding leads to the equation of a nonholonomic constraint

$$dX = ad\varphi\cos\theta, \qquad dY = ad\varphi\sin\theta \tag{1.3}$$

Let m_1, A and C be, respectively, the mass and the central moments of inertia of the wheel. Neglecting the mass of the frame we construct the Lagrangian

$$L = \tfrac{1}{2}m\,(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) + \tfrac{1}{2}m_1\,(X^{\cdot 2} + Y^{\cdot 2}) + \tfrac{1}{2}A\theta^{\cdot 2} + \tfrac{1}{2}C\varphi^{\cdot 2} - mgz$$

The equation of motion in variables φ and θ are

$$\begin{aligned}(A + m\rho^2)\,\theta^{\cdot\cdot} + ma\rho\theta^{\cdot}\varphi^{\cdot} &= 0\\ [(m + m_1)\,a^2 + mb^2 + C]\,\varphi^{\cdot\cdot} - ma\rho\theta^{\cdot 2} &= -mgb\end{aligned} \tag{1.4}$$

Following Hamel we neglect the mass of the wheel ($m_1 = A = C = 0$) and instead of (1.4) we obtain Equations

$$\rho\theta^{\cdot\cdot} + a\theta^{\cdot}\varphi^{\cdot} = 0, \qquad (a^2 + b^2)\,\varphi^{\cdot\cdot} - a\rho\theta^{\cdot 2} = -gb \tag{1.5}$$

which together with the linear nonholonomic constraints (1.3) describe the inertial motion of the considered system.

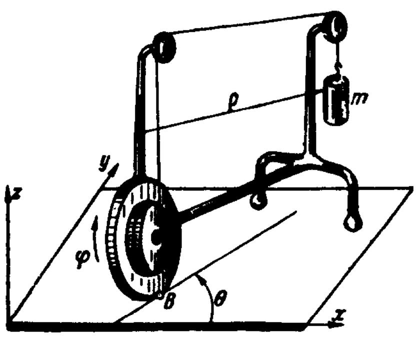

Fig. 1

Equations (1.1) to (1.5) agree with the equations derived by Hamel in [4] for the case of inertial motion. The drawings shown in Hamel's book are for the case $\rho < 0$. We shall consider separately the cases $\rho > 0$ and $\rho < 0$ The system whose motion is described by Equations (1.3) and (1.5) we shall call the nondegenerate system.

The equations of motion of a nonholonomic system with nonlinear constraints were obtained by Hemel by letting $\rho \to 0$. From (1.5) follows that when $\rho \to 0$

$$\theta^{\cdot} = 0, \quad (a^2 + b^2)\,\varphi^{\cdot\cdot} = -gb \tag{1.6}$$

Let us consider the variables x,y,z. From (1.3) and (1.2) follows that when $\rho \to 0$

$$x^{\cdot} = a\varphi^{\cdot}\cos\theta, \quad y^{\cdot} = a\varphi^{\cdot}\sin\theta \tag{1.7}$$

Eliminating from the above equations the variables φ and θ, using (1.1), and

$$x^{\cdot}y^{\cdot\cdot} - x^{\cdot\cdot}y^{\cdot} = z^{\cdot 2}\theta^{\cdot}$$

obtained from (1.7), we have the nonlinear equation of a nonholonomic constraint

$$x^{\cdot 2} + y^{\cdot 2} = (a^2 / b^2)\, z^{\cdot 2} \tag{1.8}$$

and the equation of motion (1.6) in the form

$$x^{\cdot}y^{\cdot\cdot} - x^{\cdot\cdot}y^{\cdot} = 0, \qquad (a^2 + b^2)\, z^{\cdot\cdot} = -gb^2 \tag{1.9}$$

Equations (1.9) (with the generalized forces along the x and y coordinates) were also obtained by Hamel [4]. Besides, Hamel derived these equations by Gauss's principle, which convinced him that they must be correct. Thus, the system of Appell and Hamel, with nonlinear, nonholonomic constraints, has been obtained from a nonholonomic system with linear constraints by taking the limit when $\rho \to 0$. However, taking this limit lowers the order of the system of differential equations causing its degeneration and it is not clear at the outset, whether the motions of the limiting system (1.6) and (1.7) are the same as the motions of the nondegenerate system when $\rho \to 0$. The question whether the equations of the degenerate system (1.6) and (1.7) describe correctly the motion of the initial system with the vanishingly small ρ remains open.

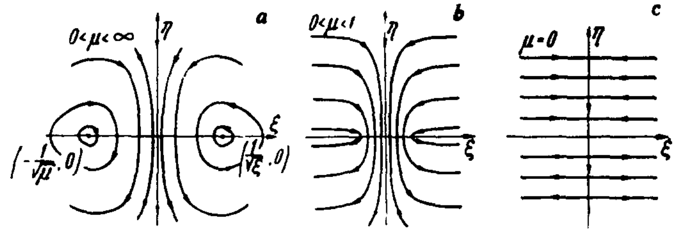

Fig. 2

In this paper we investigate the motions of a nondegenerate system at $\rho > 0$ and at $\rho < 0$, the limiting motions of a nondegenerate system when $|\rho| \to 0$, and also motions of a degenerate system, and we are able to answer the posed question.

2. Dynamics of a nondegenerate system. Let us introduce new variables ξ and η through the relations

$$\theta^{\cdot} = \alpha\xi, \qquad \varphi^{\cdot} = \beta\eta \qquad \left(\alpha = \left(1 + \frac{b^2}{a^2}\right)^{1/2}, \beta = \frac{gb}{a^2 + b^2}\right)$$

Equations of motion (1.5) will have the form

$$\mu\xi^{\cdot} = \mp\, \xi\eta, \qquad \eta^{\cdot} = -1 \pm \mu\xi^2 \qquad \left(\mu = \frac{|\rho|}{a\beta} \geqslant 0\right) \tag{2.1}$$

The signs plus or minus between terms refer to the two cases $\rho > 0$ and $\rho < 0$. The upper sign refers to $\rho > 0$, the lower to $\rho < 0$. Motions of the system considered can be represented by the displacements of the tracing point in the phase plane $\xi\eta$. Let us consider the partition of this plane by trajectories. Dividing the second equation (2.1) by the first one we obtain Equation

$$\frac{d\eta}{d\xi} = \mu \frac{-1 \pm \mu\xi^2}{\mp \xi\eta} \tag{2.2}$$

with separable variables. From it we find the family of integral curves

$$\eta^2 = \mu (\pm \ln \xi^2 - \mu\xi^2) + \text{const} \tag{2.3}$$

These curves are the trajectories which partition the $\xi\eta$ plane and are shown in Fig. 2 and 3, where the arrows indicate the direction of motion of the tracing point.

A. The case $\rho > 0$. The motion of the tracing point in the $\xi\eta$ plane is described by Equations

$$\mu\xi^{\cdot} = -\xi\eta, \qquad \eta^{\cdot} = -1 + \mu\xi^2 \tag{2.4}$$

Assigning to the arbitrary constant in Equations (2.3) the initial values $\xi = \xi_0$ and $\eta = \eta_0$, we obtain

$$\eta^2 = \mu\, [\ln (\xi^2 / \xi_0^2) - \mu\, (\xi^2 - \xi_0^2)] + \eta_0^2 \tag{2.5}$$

The phase trajectories in the $\xi\eta$ plane are closed curves, one inside another and contain the singular points $(-1 / \sqrt{\mu}, 0)$ and $(1 / \sqrt{\mu}, 0)$, which are themselves phase trajectories (Fig. 2 a).

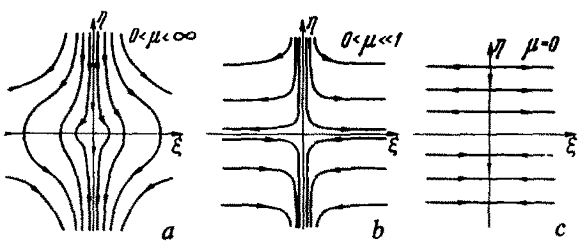

Fig. 3

Substituting (2.5) in the first equation of (2.4) we obtain

$$\mu \frac{d\xi}{dt} = \mp\, \xi \left(\mu \left[\ln \frac{\xi^2}{\xi_0^2} - \mu\, (\xi^2 - \xi_0^2)\right] + \eta_0^2\right)^{1/2}$$

where the upper sign corresponds to the motion of the tracing point along the upper half of the closed integral curve (Fig. 2 a) and passing through the point ξ_0, η_0, and the lower sign to the motion along the lower half. The above equation determines the relationship $\xi = \xi(t)$

$$\mp \mu \int_{\xi_0}^{\xi} \frac{d\xi}{\xi \sqrt{\mu\, [\ln (\xi^2 / \xi_0^2) - \mu\, (\xi^2 - \xi_0^2)] + \eta_0^2}} = t - t_0 \tag{2.6}$$

To find the time dependence of the other variables it is sufficient to express them in terms of ξ. Thus, for the angle of rotation θ of the wheel plane we have the relation $\theta^{\cdot} = \alpha\xi$. From this we find

$$\mp \alpha\mu \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\mu [\ln(\xi^2/\xi_0^2) - \mu(\xi^2 - \xi_0^2)] + \eta_0^2}} = \theta - \theta_0 \tag{2.7}$$

The angle of spin of the wheel φ is expressed by the relation

$$\varphi^{\cdot} = \beta\eta = \pm \beta \sqrt{\mu [\ln(\xi^2/\xi_0^2) - \mu(\xi^2 - \xi_0^2)] + \eta_0^2}$$

$$(\varphi - \varphi_0 = \mp \mu\beta \ln(\xi/\xi_0)) \tag{2.8}$$

Corresponding expressions for the coordinates X and Y of the point B, which is the point of contact of the wheel with the plane, are obtained from Equations (1.3). The speed V of the point B equals (2.9)

$$V = \sqrt{\dot{X}^{2} + \dot{Y}^{2}} = a|\varphi'| = a\beta \sqrt{\mu [\ln(\xi^2/\xi_0^2) - \mu(\xi^2 - \xi_0^2)] + \eta_0^2}$$

From this follows that at the instants of time when ξ takes on the values $\xi = \xi_1$ and $\xi = \xi_0$, where ξ_1 and ξ_2 are the roots of Equation

$$\mu [\ln(\xi^2/\xi_0^2) - \mu(\xi^2 - \xi_0^2)] + \eta_0^2 = 0 \tag{2.10}$$

(expressing the condition for the intersection of the $\eta = 0$ axis by the integral curve), an instantaneous rest of the point B occurs. Since at these instants of time the angular velocity $\theta^{\cdot} \neq 0$, and φ' changes sign, the trajectory of the point B forms here a cusp with the singular point of the stationary point type. Between these singular points the trajectory is a part of a spiral.

The results obtained permit to give a qualitative description of the behavior of the system and to determine the character of a trajectory traced by the point B at various initial conditions.

A simpler kind of motion of a system is obtained in the case, when in the $\xi\eta$ plane the tracing point coincides with the singular point $(\pm \mu^{-1/2}, 0)$. In this case $\varphi^{\cdot} = 0$, the point B is at rest, and the mass m, hanging at the constant height $z = z_0$, rotates about an axis passing through B, with the constant angular velocity $\theta^{\cdot} = \alpha / \sqrt{\mu}$. Physically it means that the moment of the frictional force $F^* = m\rho\theta^{\cdot 2}$ about the center of the wheel balances the moment of the gravitational force mg, that is

$$a\rho\theta^{\cdot 2} = gb \quad \text{or} \quad \xi = \pm 1 / \sqrt{\mu}$$

Another simpler kind of motion is obtained when in the $\xi\eta$ plane the tracing point moves along the $\xi = 0$ axis. In this case $\theta = \theta_0 = \text{const}$, that is the trajectory of the point B is a straight line. From the equation of motion $\eta^{\cdot} = -1$ follows, that the mass falls down with the constant acceleration

$$z^{\cdot\cdot} = -\frac{gb^2}{a^2 + b^2} \tag{2.11}$$

All the remaining motions of the system correspond to displacements of the tracing point in the $\xi\eta$ plane along curves inclosing a region with the points $(\pm 1/\sqrt{\mu}, 0)$. Further, the angular velocity $\theta^{\cdot}$ of the rotation of the plane of the wheel oscillates about a certain average $\theta^{\cdot} =$ const, and the variable $\varphi^{\cdot}$ oscillates about its zero value. Accordingly, in one circumambulation of a tracing point along the closed curve in Fig. 2*a* the point B, which is the point of contact of the wheel with the plane, describes a curve which we shall call a cell. One of the possible cells is shown in Fig. 4*a*,

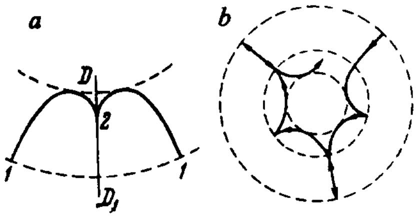

Fig. 4

where the point 1 corresponds to the value $\xi = \xi_1$, and the point 2 to the value $\xi = \xi_2$. The whole trajectory consists of identical cells (1 - 2 - 1), which arrange themselves along a strip marked in Fig. 4*a* by the dotted curve. A cell is symmetrical with respect to the line DD_1. Therefore the dotted curves can only be circular. In this way a trajectory of a point is always contained in a circular strip. Depending on the initial conditions and the relations between the parameters of the system, the trajectory of the point B is either a closed curve, or a quasi-periodic curve, filling densely everywhere the annular region.

B. The case $\rho < 0$. The equations of motion (2.1) in this case have the form

$$\mu\xi^{\cdot} = \xi\eta, \qquad \eta^{\cdot} = -1 - \mu\xi^2 \qquad \left(\mu = \frac{|\rho|}{\alpha\beta} > 0\right) \tag{2.12}$$

The phase trajectories in the $\xi\eta$ plane form open curves, symmetrical with respect to the coordinate axes (Fig. 3*a*). Let us write the equation (2.3) of the integral curve passing through the point $\xi = \xi_0$, $\eta = \eta_0$

$$\eta^2 = -\mu\,[\ln(\xi^2/\xi_0^2) + \mu(\xi^2 - \xi_0^2)] + \eta_0^2 \tag{2.13}$$

To obtain equations expressing time dependence of ξ and of other variables of interest, we can proceed the same way as in the case $\rho > 0$. However, we can easily find out that all these equations can be obtained also from (2.6) to (2.9) by replacing μ by $-\mu$. Equation (2.10) changes into

$$\mu\,[\ln(\xi^2/\xi_0^2) + \mu(\xi^2 - \xi_0^2)] = \eta_0^2 \tag{2.14}$$

which has only one root $\xi^2 = (\xi^2)_0$. From (2.9) and from Fig. 3*a* follows that depending on the initial conditions, the trajectory of the point B has only one cusp-like singularity when $\eta_0 > 0$, and none when $\eta_0 < 0$. Since the infinite branches of all the integral curves in Fig. 3*a* approach the $\xi = 0$ axis, the trajectories of the point B when $t \to -\infty$ and $t \to +\infty$ also have infinite branches which approach asymptotes. Consequently, the whole $(-\infty < t < +\infty)$ trajectory of the point B has the form shown in Fig. 5*a*. Thus the behavior of the system in the case $\rho < 0$ differs essentially from its behavior in the case $\rho > 0$. Only for the special value of the initial conditions, when $\xi_0 = 0$, the trajectory of the point B can be a straight line in both cases and the behavior of the system identical.

3. Motion of the degenerate system. Limiting motion of the nondegenerate system when $|\rho| \to 0$. We have shown in Section 1 that the equations of motion (1.6) or (1.9) of a system with nonlinear, nonholonomic constraint (1.8) are obtained from Equations (1.5) or (2.1) of the nondegenerate system when $\rho \to 0$. In this case Equations (2.1) take the form

$$\xi^{\cdot} = 0, \qquad \eta^{\cdot} = -1 \qquad (3.1)$$

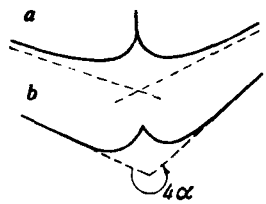

Fig. 5

From this follows that the phase space of the degenerate system is the straight line $\xi = 0$. With arbitrary initial conditions the displacement of the tracing point along the phase line $\xi = 0$ with the constant velocity $\eta^{\cdot} = -1$ corresponds to the inertial motion of a system. This means that the trajectory of the point of contact B , where the wheel touches the plane, will be always a straight line along which the wheel rolls with constant acceleration $\varphi^{\cdot\cdot} = -\beta$. We will show now that the motion described by equations (3.1) differs from the limiting motion, which the system will perform when the quantity $|\rho|$ tends to zero. For this purpose it is sufficient to investigate a motion of the nondegenerate system when $\rho \to \pm 0$. As we have done previously we shall consider separately the cases $\rho > 0$ and $\rho < 0$

a) The case $\rho > 0$. From the first equation (2.4) follows that

$$\lim_{\mu\to 0} \xi^{\cdot} = -\lim_{\mu\to 0} \frac{\xi\eta}{\mu} = \begin{cases} -\infty, & \text{when } \xi\eta > 0 \\ +\infty, & \text{when } \xi\eta < 0 \end{cases} \qquad (\xi \neq 0,\ \eta \neq 0)$$

that is the rate of change of the coordinate ξ in the $\xi\eta$ plane when $\mu \to 0$ grows without bounds at all values of $\xi \neq 0$ and $\eta \neq 0$. From this and from Equations (2.2) and (2.5) follows that when μ approaches zero the picture of the partition of the phase plane by the trajectories changes as shown in Fig. 2b and 2c. In the limiting cases ($\mu = 0$) the whole phase plane $\xi\eta$ is the region of fast motions, with the exception of the line $\xi = 0$ which is the axis of slow motions. Besides, in the region $\eta > 0$ the axis of slow motions $\xi = 0$ is stable with respect to the fast motions, and in $\eta < 0$ it is unstable.

We shall find the limiting motion which is being approached by the motion of the nondegenerate system when $\mu \to 0$. For this purpose we shall consider the motion of the tracing point in the $\xi\eta$ plane along one of the closed integral curves. The limiting position of an integral curve is shown in Fig. 6. Indeed, from Equation (2.5) follows that when $\mu \to 0$ this equation degenerates into the pair of straight lines $\eta = \pm \eta_0$. Besides, the roots ξ_1 and ξ_2 of the equation (2.10) approach, respectively, zero or infinity by the formula

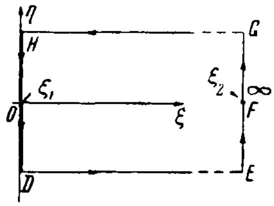

Fig. 6

$$|\xi_1| \approx \exp\frac{-\eta_0^2}{2\mu}, \quad |\xi_2| \approx \frac{\eta^\circ}{\mu} \qquad (3.2)$$

We shall find the intervals of variation of time t, and also of the angles θ and φ when the tracing point moves on the segments OD, DE and EF in Fig. 6. The quantities which refer to these three segments will have indices 1, 2, 3 respectively.

On the segment OD $\xi = 0$, hence by (2.4) follows

$$\int_0^{\eta_0} d\eta = -\int_0^{t_1} dt, \quad \text{or} \quad t_1 = \eta_0$$

the derivative $\theta^\cdot = \alpha\xi$, therefore $\theta_1 = 0$

To calculate the quantity φ_1, we shall use the integral relation $\varphi = {}^1/_2\beta(\eta^2 + \mu^2\xi^2) + \text{const}$; from which $\varphi_1 = -{}^1/_2\beta\eta_0^2$.

On the segment DE. From (2.6) we find

$$t_2 = \lim_{\mu\to 0} \frac{\mu}{\eta_0} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\xi} = \lim_{\mu\to 0} \frac{\mu}{\eta}\left[\ln\frac{\eta_0}{\mu} + \frac{\eta_0^2}{2\mu}\right] = \frac{1}{2}\,\eta_0$$

From the relation $\theta^\cdot = \alpha\xi$ and from the first equation in (2.4) we obtain

$$\theta_2 = \alpha\int_0^{t_2} \xi dt = \lim_{\mu\to 0}\frac{\alpha\mu}{\eta_0}\int_{\xi_1}^{\xi_2} d\xi = \alpha$$

And finally

$$\varphi_2 = -\beta\eta_0\int_0^{t_2} dt = -\frac{1}{2}\beta\eta_0^2$$

On the segment EF. From the second equation in (2.4) we find

$$\int_{-\eta_0}^{0} d\eta = \frac{\eta_0^2}{2\mu}\int_0^{t_3} dt, \quad \text{or} \quad t_3 = \lim_{\mu\to 0}\frac{\mu}{\eta_0} = 0$$

Further

$$\theta_3 = \lim_{\mu\to 0}\alpha\int_0^{t_3}\xi dt = \lim_{\mu\to 0}\alpha\frac{\eta_0}{\mu}\frac{\mu}{\eta_0} = \alpha, \qquad \varphi_3 = \lim_{\mu\to 0}\beta\,(\eta)_{cp}\frac{\mu}{\eta_0} = 0$$

On the segments FG, GH, HO the motion is symmetric.

The results obtained permit to display the form of the trajectory of the point of contact of the wheel with the plane B, in the limiting case when $\mu = 0$, and also the character of the limiting motion of the system.

When the tracing point moves along the contour *ODEFGHO* (Fig. 6) then the system moves in the following way: in the interval of time $t_1 = \eta_0$ the wheel rolls through the angle $\varphi_1 = -{}^1/_2\beta\eta_0{}^2$, moving along the straight line $(\theta_1 = 0)$, then in the interval of time $t_2 = \frac{1}{2}\eta_0$ the wheel rolls through the same angle, that is $\varphi_2 = \varphi_1 = -{}^1/_2\beta\eta_0{}^2$, and the plane of the wheel rotates through the angle $\theta_2 = \alpha$.

The trajectory of the point B on this segment has a form of an arc, whose length equals the preceding straight line segment. Further, the plane of the wheel rotates instantaneously through the angle $\theta_3 + \theta_4 = 2\alpha$ and the spin of the wheel changes direction. The trajectory of the point B is then at the corner point where the angle $\theta = 2\alpha$. Then the wheel again describes an arc-like trajectory, rolling in the interval of time $t_5 = \frac{1}{2}\eta_0$ through the angle $\varphi_5 = {}^1/_2\beta\eta_0{}^2$, and its plane through the angle $\theta_5 = \alpha$.

Finally, in the interval of time $t_6 = \eta_0$ the wheel moves on a straight line, rolling through the angle $\varphi_6 = {}^1/_2\beta\eta_0{}^2$. At the end of this segment of the trajectory the spin of the wheel changes direction, and the motion repeats itself.

The whole trajectory will have the form of a rosette, which can be a closed or an open curve filling densely everywhere a certain annular region.

The size of this region is proportional to the magnitude of the initial value $|\eta_0|$ of the velocity of spin of the wheel. The form of a trajectory of the limiting motion is shown in Fig. 4 *b* .

b) T h e c a s e $\rho < 0$. In this case from the first equation in (2.12) follows that

$$\lim_{\mu\to 0} \xi^{\cdot} = \lim_{\mu\to 0} \frac{\xi\eta}{\mu} = \begin{cases} +\infty, & \text{when } \xi\eta > 0 \\ -\infty, & \text{when } \xi\eta < 0 \end{cases} \quad \begin{pmatrix} \xi \neq 0 \\ \eta \neq 0 \end{pmatrix}$$

When $\mu \to 0$ the picture of the partition of the phase plane by the trajectories changes according to Fig. 3*b* and 3*c* .

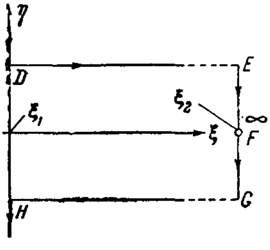

Fig. 7

In the limiting case $(\mu = 0)$ the whole phase plane will be also the region of fast motions, and the $\xi = 0$ axis will be the region of slow motions. Unlike in the case $\rho > 0$, the semi-axis $\xi = 0$, $\eta > 0$ is now unstable with respect to the fast motions, and the semi-axis $\xi = 0$, $\eta < 0$ is stable. To derive the limiting motion of this nondegenerate system when $\mu \to 0$, we shall consider the displacement of the tracing point along the the integral curve, limiting position of which is shown in Fig. 7, where the value of ξ_1 and ξ_2 are found, as in previous cases, from (3.2). The intervals of variation of the time t , and also of the angles θ

and φ when the tracing point (Fig. 7) moves along the segments DE, EF, FG and GH are found as in the case $\rho > 0$.

The character of the limiting motion of the system and the form of the trajectory of the point B will be as follows: when the tracing point moves along the $\xi = 0$ axis up to the point D , the wheel moves along a straight line with a constant acceleration. After that, in the interval of time $t_1 = \frac{1}{2}\eta_0$ the wheel rolls through the angle $\varphi_1 = \frac{1}{2}\beta\eta_0^2$, and its plane rotates through the angle $\theta_1 = \alpha$. On this segment the trajectory of the point B has the form of an arc. Further, the plane of the wheel rotates instantaneously through the angle $\theta_2 + \theta_3 = 2\alpha$, and the spin changes direction. The trajectory of the point B will coincide then with the corner point. Then the wheel describes again an arc, rolling in the interval of time $t_4 = \frac{1}{2}\eta_0$ through the angle $\varphi_4 = -\frac{1}{2}\beta\eta_0^2$, and its plane rotating through the angle $\theta_4 = \alpha$. After this the wheel moves along a straight line, forming with the initial straight line the angle $\theta = 4\alpha$. The form of the limiting trajectory in the case $\rho < 0$ is shown in Fig. 5*b* .

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